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Closed string amplitudes in open string field theory

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Abstract: We investigate gauge invariant operators corresponding to on-shell closed string states in open string field theory. Using both oscillator representation and conformal mapping techniques, we calculate a two closed string tachyon amplitude that connects two gauge invariant operators by an open string propagator. We find that this amplitude is in a complete agreement with the usual disc amplitude.

Keywords: Bosonic Strings, String Field Theory.

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Contents

1. Introduction

String field theory is a fascinating formulation to investigate non-perturbative aspects of string theory. The progress in the last three years in string field theory [1]–[5], which has been brought by calculational techniques of level truncation scheme [6], enables us to describe the decay process of D-branes in terms of classical solutions in the theory.

One of the most interesting facts in open string field theory is that, as pointed out at the early stage formulating covariant string field theories [7]–[10], closed strings are concealed in the theory as a bound state of the open string, because open string loop diagrams look like closed string propagations if we change time-slice. Indeed we can see a tower of closed string poles explicitly in non-planar one loop amplitudes [11]. While there are some recent works related to closed strings in open string field theory [12, 13], the nature of closed strings in the theory is not fully clear. In particular, we can not identify closed strings with asymptotic states in open string field theory, though such states are required by unitarity of the theory. If we introduce on-shell closed strings as gauge invariant operators in open string field theory, the theory generates a single cover of moduli space of Riemann surfaces with punctures and boundaries [14]. However, we can not produce purely closed string amplitudes in the theory.

In order to describe purely closed string processes, an interesting proposal was made in the papers [15, 16] by Drukker. If we obtain the tachyon vacuum solution and the theory expanded around the solution, we may reproduce closed string amplitudes by the Feynman rules of the theory with the gauge invariant operators as sources. Because the propagator in the theory around the solution does not generate any propagation of boundaries of strings. For this scenario, a possible candidate for the solution was constructed in ref. [17] and some desirable results were obtained in refs. [18]–[20].

In this paper we investigate the gauge invariant operators corresponding to on-shell closed string states in order to develop techniques for describing purely closed string processes. Intuitively we expect that the operators can be constructed by locating a closed string vertex operator at the midpoint on the identity string field. There are, however, some subtleties that ghost operators in the vertex operators diverge at the midpoint. To avoid this difficulty, a regularization method is proposed in ref. [22]. We show that this regularized gauge invariant operator agrees with the operator defined by the method of conformal field theory.

In section 2, we define the gauge invariant operators making use of conformal field theory. We extend the operator for the closed string tachyon given in ref. [22] to that of D-branes background. In section 3, we calculate a two closed string tachyon amplitude using the Feynman rules and the gauge invariant operator. Mapping the world sheet into a whole complex plane explicitly, we show that the amplitude derived from the gauge invariant operator agrees with the usual disc amplitude. Moreover, we calculate the amplitude using the operator formalism and we find that the result agrees with the same amplitude. We give summary and discussion in section 4.

2. Closed string source terms

Open string field theory with a midpoint interaction is defined by the action [9]

$$S_{\rm o} = \int \left(\Psi * Q_{\rm B}\Psi + \frac{2}{3}\Psi * \Psi * \Psi \right). \tag{2.1}$$

We can incorporate closed strings in the theory by adding a source term to the action [14, 21, 22]:

$$S = S_{\rm o} + \int \Phi\Psi\,, \tag{2.2}$$

where Φ corresponds to an on-shell closed string state. The Feynman rules derived from the action provide a single cover of moduli space of open and closed strings amplitudes [14].

Intuitively, we expect that this source term is represented by a closed string vertex operator on the identity string field [21, 22]:

$$\int \Phi\Psi = \langle I|\, V\left(\frac{\pi}{2}\right) |\Psi\rangle\ , \tag{2.3}$$

where $V(\sigma)$ is a closed string vertex operator

$$V(\sigma) = c_+(\sigma)c_-(\sigma)O(\sigma)\,. \tag{2.4}$$

However, the midpoint ghost insertion has a singularity on the identity and then we must regularize this expression to be well-defined. In ref. [22], it is proposed that a regulated source term should be defined as

$$\int \Phi\Psi = \langle \tilde{I}|\, O\left(\frac{\pi}{2}\right) |\Psi\rangle\ , \tag{2.5}$$

where $\tilde{I}$ denotes the state in which the anti-ghost factors of the identity state are removed. In this definition, we do not include the infinite factor arising from the commutator of the ghost and anti-ghost fields.

If faithful to ref. [14], we should define closed string sources in terms of conformal field theory. First we define the state $\langle V|$ for the closed vertex operator $V(\sigma)$ as

$$\langle V|\,|A\rangle = \left\langle h\left[V\left(\frac{\pi}{2}\right)\,A(0)\right]\right\rangle, \tag{2.6}$$

where the state $|A\rangle$ is created by an operator as $|A\rangle = A(w=0)\,|0\rangle$ and h denotes the conformal transformation induced by the function $h(w) = 2w/(1-w^2)$, which maps a unit half disc to an upper half plane [24], and $\langle\cdots\rangle$ denotes correlation function on the upper half plane.[1] Using this state, we should define the closed string source term as

$$\int \Phi * \Psi = \langle V|\,|\Psi\rangle\ . \tag{2.7}$$

By definition, it is easily seen that this source term is a gauge invariant operator. Moreover, it is known that a correct amplitude is reproduced by level truncation analysis based on this definition [23].

From (2.6), the gauge invariant operator for a closed string tachyon is defined as

$$\langle T;\,p,\,k|\,|A\rangle = i^{-1}\,\langle\, c(i)\,c(-i)\,e^{ip\cdot X(i,-i)}\,e^{ik\cdot X(i,-i)}\,h[A(w=0)]\,\rangle\,, \tag{2.8}$$

where the correlation function is defined on the whole complex plane using the standard doubling trick. Here, we consider string field theory on a Dp brane and then we impose Neumann boundary conditions to $X^\mu(z,\bar z)$ $(\mu = 0,\ldots,p)$ and Dirichlet boundary conditions to $X^i(z,\bar z)$ $(i = p+1,\ldots,25)$. p^μ and k^i denote the momenta in tangential and orthogonal directions to a Dp brane, respectively.

From the definition (2.8), the oscillator expression of the gauge invariant operators for a closed string tachyon is given by

$$\begin{aligned}\langle T;\,p,\,k|\,|\Psi\rangle \;=\; \langle -p|\,c_{-1}c_0\,\exp\Bigg[&(\alpha' p^2 - 1)\log 4 + \sum_{n=1}^{\infty}(-1)^n\,\times\\ &\times\left\{-\frac{1}{2n}\alpha_n\cdot\alpha_n - c_n b_n - \frac{\sqrt{2\alpha'}}{n}p\cdot\alpha_{2n} -\right.\\ &\left.-\frac{2i\sqrt{2\alpha'}}{2n-1}k\cdot\alpha_{2n-1}\right\}\Bigg]\,|\Psi\rangle\ .\end{aligned} \tag{2.9}$$

This is an extended expression from that of ref. [22] to the case that contains Dirichlet directions. The detail derivation of this expression will appear in a forthcoming paper [25]. Here, we only comment on the BRS invariance of this state. Though the BRS invariance is manifest by the definition (2.8), we can check it explicitly by the oscillator expression. From (2.9), we find

$$K_{2N}^{\rm matt}\,|T;\,p,\,k\rangle = 4\alpha'(p^2+k^2)(-1)^N N\,|T;\,p,\,k\rangle\ , \tag{2.10}$$

where the operator $K_{2N}^{\rm matt}$ is defined as $K_{2N}^{\rm matt} = L_{2N}^{\rm matt} - L_{-2N}^{\rm matt}$ using the matter Virasoro generators $L_n^{\rm matt}$. Similarly we obtain $K_{2N-1}^{\rm matt}\,|T;\,p,\,k\rangle = 0$. Therefore using the method in refs. [26, 22], we conclude that, if we impose the on-shell condition $p^2+k^2 = 4/\alpha'$, the state (2.9) is BRS invariant even for the case containing the Dirichlet directions.

[1]The midpoint $\pi/2$ corresponds to i in the complex w plane.

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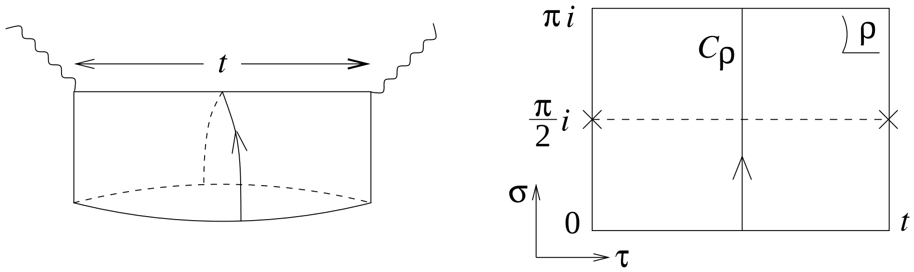

Figure 1: The world sheet corresponding to the two closed tachyon amplitude.

3. Closed string tachyon amplitudes

We consider the closed tachyon amplitude derived from the source term (2.8). The simplest example is derived from the diagram connecting two tachyon sources by an open string propagator:

$$(2\pi)^{p+1}\,\delta^{p+1}(p+p')\,\mathcal{A} = \int_0^\infty dt\,{}_1\langle T;\,p,\,k\,|_2\langle T;\,p',\,k'\,|\,b_0^{(1)}\,e^{-tL_0^{(1)}}\,|R\rangle_{12}\,. \qquad (3.1)$$

Gluing the two surface states by an open string, we find the world sheet corresponding to the integrand in this amplitude as depicted in figure 1. We introduce the coordinate $\rho = \tau + i\sigma$ into this world sheet: the tachyon vertices insert on $\rho = \pi\, i/2$ and $t + \pi\, i/2$, and the points σ and $\pi - \sigma$ are identified with each other at the boundaries $\tau = 0,\, t$. Consequently, we can rewrite the amplitude as

$$\begin{aligned}\mathcal{A} = \int_0^\infty dt \int_{C_\rho} \frac{d\rho}{2\pi i} &\left\langle b(\rho)\, c\left(\frac{\pi i}{2}\right) c\left(\frac{-\pi i}{2}\right) c\left(t+\frac{\pi i}{2}\right) c\left(t-\frac{\pi i}{2}\right)\right\rangle_\rho \times \qquad (3.2)\\ &\times\langle\, e^{ip\cdot X(\pi i/2,\,-\pi i/2)} e^{ip'\cdot X(t+\pi i/2,\,t-\pi i/2)}\,\rangle_\rho\,\langle\, e^{ik\cdot X(\pi i/2,\,-\pi i/2)} e^{ik'\cdot X(t+\pi i/2,\,t-\pi i/2)}\,\rangle_\rho\,,\end{aligned}$$

where we use the doubling trick and the contour C_ρ is a path from $\sigma = -\pi i$ to $\sigma = \pi i$ at an arbitrary τ, and $\langle\cdots\rangle_\rho$ denotes correlation function on the ρ plane.

Since the ρ plane is topologically equivalent to a disc or after the doubling trick to a sphere, it is more convenient to map the ρ plane to an upper half or a whole complex plane. First we rotate the ρ plane 90 degrees and put it on a region in the u plane as depicted in figure 2. In the u plane the left (right) region of the imaginary axis corresponds to the holomorphic (anti-holomorphic) part of the open string, and the closed tachyon vertices are inserted at the points C, D and their anti-holomorphic counter-points. We introduce the parameter τ as $\pi\tau/2 = it$.

We can transform this region in the u plane to a whole complex z plane by the mapping

$$z = -i\,\frac{\vartheta_1(u|\tau)}{\vartheta_4(u|\tau)}\,, \qquad (3.3)$$

where $\theta_i(u|\tau)$ $(i = 1,2,3,4)$ are theta functions with periods 1 and τ [27]. Using properties of theta functions, we find the relations $z(-\pi-u) = z(u)$, $z(\pi-u) = z(u)$ and $z(0) = z(\pm\pi) = 0$, and also that z is imaginary if u is real. Therefore we find a cut from $z(-\pi/2)$

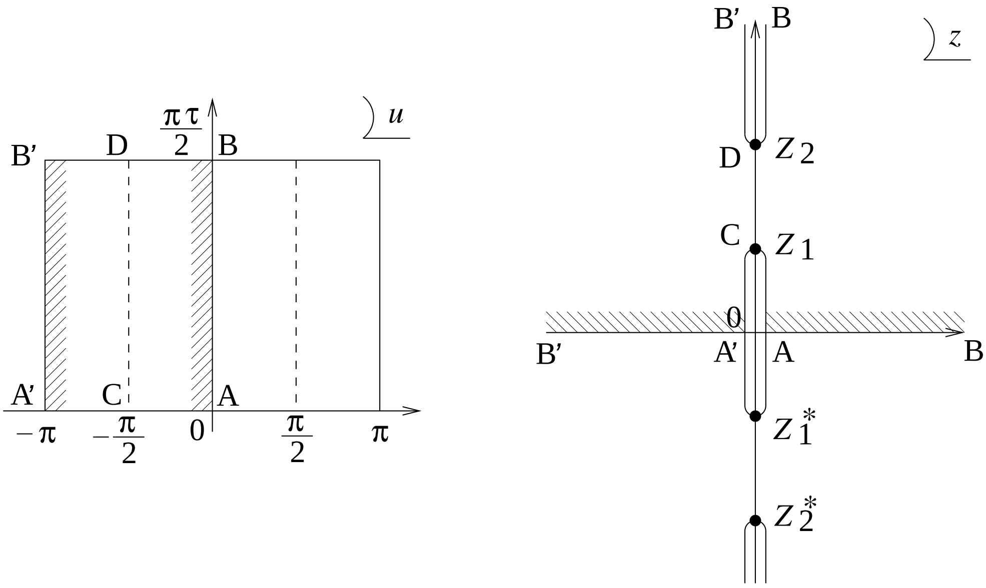

Figure 2: The u and z planes for the two closed tachyon amplitude.

to $z(\pi/2)$ in the z plane and this cut corresponds to the identification of AC and A$'$C in the u plane. The points $z(\pm\pi/2)$ are the insertion points of the closed tachyon vertices in the z plane and they are expressed as

$$
\begin{aligned}
z\left(\frac{-\pi}{2}\right) &= -i\,\frac{\vartheta_1(-\pi/2\,|\tau)}{\vartheta_4(-\pi/2\,|\tau)} = i\,\frac{\vartheta_2}{\vartheta_3} = Z_1\,, \\
z\left(\frac{\pi}{2}\right) &= -i\,\frac{\vartheta_1(\pi/2\,|\tau)}{\vartheta_4(\pi/2\,|\tau)} = -i\,\frac{\vartheta_2}{\vartheta_3} = Z_1^*\,,
\end{aligned}
\tag{3.4}
$$

where ϑ_i denote $\vartheta_i(0|\tau)$ [27]. Moreover we can derive the relation $z(u+\pi\tau/2) = -1/z(u)$ and then we find another cut from the points $Z_2 = i\vartheta_3/\vartheta_2$ and $Z_2^* = -i\vartheta_3/\vartheta_2$ to the infinity along the imaginary axis. We can see also that the open string boundary in the u plane corresponds to the real axis in the z plane.

The function $z(u)$ satisfies [27]

$$
\left(\frac{dz}{du}\right)^2 = -(\vartheta_2^2 + z^2\,\vartheta_3^2)(\vartheta_3^2 + z^2\,\vartheta_2^2)\,. \tag{3.5}
$$

This differential equation implies that the function $z(u)$ is an elliptic function with the branch points $\pm i\,\vartheta_2/\vartheta_3$ and $\pm i\,\vartheta_3/\vartheta_2$ in the z plane [27]. Indeed the function $z(u)$ is an doubly-periodic function with periods 2π and $\pi\tau$. Since the domain now considered in the u plane is half of the fundamental parallelogram, it is clear that the domain is mapped to half domain of two sheets, in this case the whole complex plane. In addition, from the differential equation, it follows that $dz/du \sim \sqrt{z-Z}$ near $z = Z = Z_i = Z_i^*$. Then, if z rotates by 2π around Z, u rotates by π around its corresponding point, which is a conical singularity in the u plane and the vertex insertion point [28]. Thus we can map the domain in figure 2 to the complex plane.

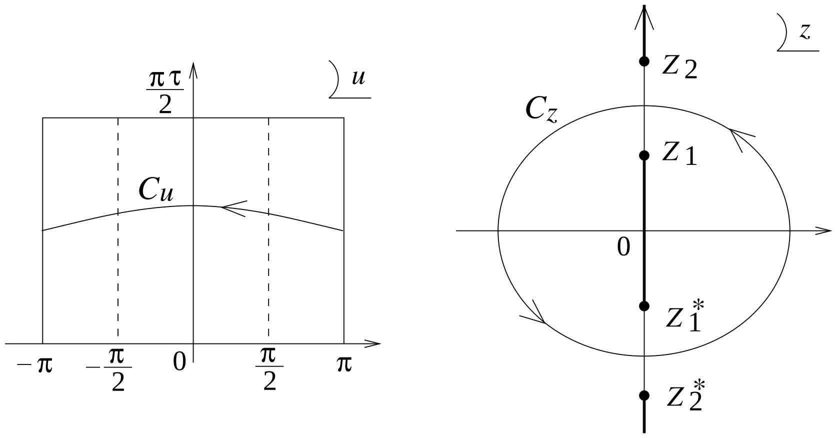

Figure 3: The contour C_z in the equation (3.6)

Now that the conformal mapping is established, we can calculate the amplitude in the z plane. By the mapping $z(u)$ the amplitude (3.2) is transformed to

$$\begin{aligned}\mathcal{A} = &\int_0^1 dx \left(\frac{dx}{dt}\right)^{-1} \oint_{C_z} \frac{dz}{2\pi}\frac{dz}{du} \langle\, b(z)c(Z_1)c(Z_1^*)c(Z_2)c(Z_2^*)\,\rangle \times \\ &\times \langle\, e^{ip\cdot X(Z_1,Z_1^*)}\, e^{ip'\cdot X(Z_2,Z_2^*)}\,\rangle \langle\, e^{ik\cdot X(Z_1,Z_1^*)}\, e^{ik'\cdot X(Z_2,Z_2^*)}\,\rangle\,, \end{aligned} \tag{3.6}$$

where conformal factors cancel using the on-shell condition, and the contour C_z encircles the points Z_1 and Z_1^* as depicted in Fig. 3. The modular parameter is given by $x = \vartheta_2/\vartheta_3$ and then the vertex insertion points are written as $Z_1 = ix$, $Z_2 = i/x$, $Z_1^* = -ix$ and $Z_2^* = -i/x$. As t changes from 0 to ∞, x runs from 1 to 0.

The matter correlation function is calculated as

$$\begin{aligned}\left\langle\, e^{ip\cdot X(Z_1,Z_1^*)} e^{ip'\cdot X(Z_2,Z_2^*)}\,\right\rangle &= 2^{-\alpha' s}\left(\frac{x^2}{1-x^4}\right)^{-\alpha' s}\,, \\ \left\langle\, e^{ik\cdot X(Z_1,Z_1^*)} e^{ik'\cdot X(Z_2,Z_2^*)}\,\right\rangle &= 2^{-4-\alpha' s}\left(\frac{1-x^2}{1+x^2}\right)^{-\alpha'(s+t/2)-4}\,, \end{aligned} \tag{3.7}$$

where the string coordinates of the Neumann directions are given by $X^\mu(z,z') = (X^\mu(z) + X^\mu(\bar{z}))/2$ and that of the Dirichlet directions are given by $X^i(z,z') = (X^i(z) - X^i(\bar{z}))/2$. The correlation functions are calculated in terms of the propagators

$$\begin{aligned}\langle\, X^\mu(z)\, X^\nu(z')\,\rangle &= -2\alpha'\eta^{\mu\nu}\,\log(z-z')\,, \\ \langle\, X^i(z)\, X^j(z')\,\rangle &= -2\alpha'\delta^{ij}\,\log(z-z')\,. \end{aligned} \tag{3.8}$$

Here, we have defined kinematical invariants as $s = -p^2 = -p'^2$ and $t = -(k+k')^2$, and we have used the on-shell condition and the momentum conservation $p^\mu + p'^\mu = 0$.

The ghost correlation function is obtained as

$$\begin{aligned}\langle\, b(z)c(Z_1)c(Z_1^*)c(Z_2)c(Z_2^*)\,\rangle &= -\frac{4(1-x^4)^2}{x^4(z^2+x^2)(z^2+x^{-2})} \\ &= 4\,\vartheta_3^4\,\frac{(1-x^4)^2}{x^2}\left(\frac{dz}{du}\right)^{-2}\,, \end{aligned} \tag{3.9}$$

where we have used the differential equation (3.5). Using this result, we can perform the contour integration in (3.6):

$$\begin{aligned}\oint_{C_z} \frac{dz}{2\pi i}\frac{dz}{du}\,\langle\, b(z)c(Z_1)c(Z_1^*)c(Z_2)c(Z_2^*)\,\rangle &= \oint_{C_z}\frac{dz}{2\pi}\frac{dz}{du}\,4\,\vartheta_3^4\,\frac{(1-x^4)^2}{x^2}\left(\frac{dz}{du}\right)^{-2}\\ &= \oint_{C_u}\frac{du}{2\pi}\,4\,\vartheta_3^4\,\frac{(1-x^4)^2}{x^2}\\ &= -4\,\vartheta_3^4\,\frac{(1-x^4)^2}{x^2}\,. \end{aligned} \tag{3.10}$$

Next we evaluate dx/dt, which appears when we change the moduli parameter from t to x. The parameter $x=\vartheta_2/\vartheta_3$ is given by $x=i\,z(\pi/2)$. Then we find

$$\frac{dx}{d\tau}=\frac{\pi}{4}\frac{d^2z}{du^2}\bigg|_{u=\pi/2}\,, \tag{3.11}$$

where we have used the equations

$$\frac{\partial\vartheta_i(u|\tau)}{\partial\tau}=-\frac{i\pi}{4}\frac{\partial^2\vartheta_i(u|\tau)}{\partial u^2}\,, \tag{3.12}$$

and $\vartheta_1'(\pi/2)=\vartheta_4'(\pi/2)=0$.[2] On the other hand, differentiating both sides of eq. (3.5) with respect to u, we obtain

$$\frac{d^2z}{du^2}=-z(\vartheta_2^4+\vartheta_3^4+2\,\vartheta_2^2\,\vartheta_3^2\,z^2)\,, \tag{3.13}$$

and, if $u=\pi/2$, we find

$$\frac{d^2z}{du^2}\bigg|_{u=\pi/2}=i\,\vartheta_3^4\,x\,(1-x^4)\,. \tag{3.14}$$

From eqs. (3.11) and (3.14), it follows that

$$\frac{dx}{dt}=-\frac{\vartheta_3^4}{2}\,x\,(1-x^4)\,. \tag{3.15}$$

Putting eqs. (3.7), (3.10) and (3.15) together, the amplitude is finally obtained as

$$\mathcal{A}=\int_0^1 dx\,\frac{1-x^4}{2x^3}\left\{\frac{4x^2}{(1+x^2)^2}\right\}^{-\alpha' s}\left(\frac{1-x^2}{1+x^2}\right)^{-\alpha' t/2-4}\,. \tag{3.16}$$

We can rewrite this result to a familiar expression by a SL$(2,\mathbb{C})$ transformation. If we consider the transformation in the z plane

$$z'=\frac{x^2-1}{x^2+1}\,\frac{z+i/x}{z-i/x}\,, \tag{3.17}$$

[2]We denote differentiations with respect to u by primes.

the vertex insertion points Z_i are transformed as $Z_1^* \to \infty$, $Z_1 \to 1$, $Z_2 \to ((1-x^2)/(1+x^2))^2$ and $Z_2^* \to 0$. Then, writing $y = ((1-x^2)/(1+x^2))^2$, we find that the result agrees with the amplitude of two closed tachyons on a disc [23, 12]:

$$\mathcal{A} = \int_0^1 dy\, y^{-\alpha' t/4-2} (1-y)^{-\alpha' s-2} = B(-\alpha' t/4 - 1,\, -\alpha' s - 1)\,. \tag{3.18}$$

Let us calculate the scattering amplitude of two closed string tachyons in terms of the oscillator expression (2.9). The amplitude can be calculated as

$$\begin{aligned}
\mathcal{A} &= \int_0^\infty dt\, 4^{-2\alpha' s-2}\, q^{-\alpha' s-1}\, \mathcal{A}_{\rm gh}\, \mathcal{A}_{\rm N}\, \mathcal{A}_{\rm D}\,, \\
\mathcal{A}_{\rm gh} &= \langle 0|\, c_{-1} \exp\left[-\sum_{n=1}^\infty (-1)^n c_n b_n q^{2n}\right] \exp\left[\sum_{n=1}^\infty (-1)^n c_{-n} b_{-n}\right] c_0 c_1 \,|0\rangle \\
\mathcal{A}_{\rm N} &= \langle 0|\, \exp\left[\sum_{n=1}^\infty (-1)^n \left(-\frac{1}{2n}\alpha_{n\mu}\alpha_n^\mu\, q^{2n} - \frac{\sqrt{2\alpha'}}{n} p_\mu \alpha_{2n}^\mu\, q^{2n}\right)\right] \times \\
&\quad \times \exp\left[\sum_{n=1}^\infty (-1)^n \left(-\frac{1}{2n}\alpha_{-n\mu}\alpha_{-n}^\mu + \frac{\sqrt{2\alpha'}}{n} p'_\mu \alpha_{-2n}^\mu\right)\right] |0\rangle \\
\mathcal{A}_{\rm D} &= \langle 0|\, \exp\left[\sum_{n=1}^\infty (-1)^n \left(-\frac{1}{2n}\alpha_{ni}\, \alpha_n^i\, q^{2n} - i\frac{2\sqrt{2\alpha'}}{2n-1} k_i\, \alpha_{2n-1}^i\, q^{2n-1}\right)\right] \times \\
&\quad \times \exp\left[\sum_{n=1}^\infty (-1)^n \left(-\frac{1}{2n}\alpha_{-ni}\, \alpha_{-n}^i - i\frac{2\sqrt{2\alpha'}}{2n-1} k'_i\, \alpha_{-2n+1}^i\right)\right] |0\rangle\,,
\end{aligned} \tag{3.19}$$

where we have introduced the parameter $q = \exp(-t)$, and $\mathcal{A}_{\rm gh}$, $\mathcal{A}_{\rm N}$ and $\mathcal{A}_{\rm D}$ denote the contributions from ghost modes, the Neumann and Dirichlet direction modes, respectively. We can perform the contraction of oscillator modes in terms of a familiar formula in refs. [7] and [29]–[33]. The results are

$$\mathcal{A}_{\rm gh} = \prod_{n=1}^\infty (1-q^{2n})\,, \tag{3.20}$$

$$\mathcal{A}_{\rm N} = \prod_{n=1}^\infty (1-q^{2n})^{-(p+1)/2} \left[\prod_{n=1}^\infty \frac{1-q^{4n}}{1-q^{4n-2}}\right]^{-4\alpha' s}\,, \tag{3.21}$$

$$\mathcal{A}_{\rm D} = \prod_{n=1}^\infty (1-q^{2n})^{-(25-p)/2} \left[\prod_{n=1}^\infty \frac{1+q^{2n}}{1-q^{2n}}\right]^{-4\alpha' s-16} \left[\prod_{n=1}^\infty \frac{1+q^{2n-1}}{1-q^{2n-1}}\right]^{4\alpha'(s+t/2)+16} \tag{3.22}$$

Substituting these results into eq. (3.19), we obtain the two closed tachyon amplitude as

$$\begin{aligned}
\mathcal{A} &= \int_0^\infty dt\, \frac{1}{16q} \prod_{n=1}^\infty (1-q^{2n})^{-12} \prod_{n=1}^\infty \left\{\frac{(1-q^{2n})}{(1+q^{2n})}\right\}^{16} \times \\
&\quad \times \left[16q \prod_{n=1}^\infty \left\{\frac{(1-q^{4n})(1+q^{2n})(1-q^{2n-1})}{(1-q^{4n-2})(1-q^{2n})(1+q^{2n-1})}\right\}^4\right]^{-\alpha' s} \times \\
&\quad \times \left[\prod_{n=1}^\infty \frac{1-q^{2n-1}}{1+q^{2n-1}}\right]^{-2\alpha' t-16}\,.
\end{aligned} \tag{3.23}$$

Finally, we show that two expressions (3.16) and (3.23) of the amplitude are equivalent to each other. To exhibit the dependence of ϑ_i on the parameter q, we write it $\vartheta_i(q^2)$. Using the formulas [27]

$$\begin{aligned}\vartheta_3\left(q^{1/2}\right) &= \vartheta_3\left(q^2\right)+\vartheta_2\left(q^2\right), \qquad \vartheta_4\left(q^{1/2}\right) = \vartheta_3\left(q^2\right)-\vartheta_2\left(q^2\right), \\ \vartheta_4\left(q^2\right)^4 &= \vartheta_3\left(q^2\right)^4-\vartheta_2\left(q^2\right)^4, \end{aligned} \tag{3.24}$$

we find the equation

$$\frac{1-x^2}{1+x^2} = \frac{\vartheta_3\left(q^{1/2}\right)^2\vartheta_4\left(q^{1/2}\right)^2}{\vartheta_4\left(q^2\right)^4}. \tag{3.25}$$

We can express ϑ_i by infinite products;

$$\begin{aligned}\vartheta_3\left(q^{1/2}\right) &= G\left(q^{1/2}\right)\prod_{n=1}^{\infty}\left(1+q^{(2n-1)/2}\right)^2, \\ \vartheta_4\left(q^{1/2}\right) &= G\left(q^{1/2}\right)\prod_{n=1}^{\infty}\left(1-q^{(2n-1)/2}\right)^2, \\ \vartheta_4\left(q^2\right) &= G\left(q^2\right)\prod_{n=1}^{\infty}\left(1-q^{4n-2}\right)^2, \end{aligned} \tag{3.26}$$

where $G(q)=\prod(1-q^{2n})$. Substituting these expressions into eq. (3.25), we obtain the equation

$$\frac{1-x^2}{1+x^2} = \left[\prod_{n=1}^{\infty}\frac{1-q^{2n-1}}{1+q^{2n-1}}\right]^4. \tag{3.27}$$

Similarly, using formulas related to the theta functions, we can find the following equations

$$\begin{aligned}-\frac{dx}{dt}\frac{1-x^4}{2x^3} &= \frac{1}{16q}\prod_{n=1}^{\infty}(1-q^{2n})^{-12}\prod_{n=1}^{\infty}\left\{\frac{1-q^{2n}}{1+q^{2n}}\right\}^{16}, \\ \frac{4x^2}{(1+x^2)^2} &= 16q\left\{\prod_{n=1}^{\infty}\frac{(1-q^{4n})(1+q^{2n})(1-q^{2n-1})}{(1-q^{4n-2})(1-q^{2n})(1+q^{2n-1})}\right\}^4. \end{aligned} \tag{3.28}$$

From Eqs. (3.27), (3.28) and (3.28), we can see that the amplitude (3.16) derived from the method of conformal field theory completely agrees with the expression (3.23) obtained by the oscillator representation.

4. Summary and discussion

We have studied the gauge invariant operators in open string field theory. We have defined the gauge invariant operators in terms of conformal field theory and have shown the oscillator expression of the operator corresponding to the on-shell closed string tachyon state. Using this operator, we have calculated the two closed string tachyon amplitude on a disc in terms of conformal techniques. The resulting amplitude agrees with the usual disc amplitude. Moreover, we have calculated the amplitude in terms of the oscillator representation of the operator. We have found that both derivations lead us to the exact same amplitude. This coincidence indicates that the gauge invariant operator has a well-defined oscillator expression, although some subtleties often arise in the identity string field.

The gauge invariant operators discussed in this paper are also observables in the theory expanded around universal solutions in ref. [17], as pointed out by Drukker [16]. The propagator in the theory around a non-trivial universal solution is given by $L = L'_0/2 - (L'_2 + L'_{-2})/4 + 3/2$, where L'_n denote twisted ghost Virasoro generators [20]. We can see that this propagator generates closed surfaces and then we can regard it a kind of closed string propagators [16]. Therefore we could derive the two closed string tachyon amplitude on the sphere from the 'string diagram' connecting the gauge invariant operators by the propagator in the theory expanded around the universal solution.

We expect that the non-trivial universal solution can be identified with the tachyon vacuum solution. Because we have already proved the no open string theorem which states that there is no open string excitation perturbatively in the theory expanded around the solution [18, 20]. Another reason is that there is no stable non-perturbative vacuum in the theory around the solution and this fact implies that the theory is on the tachyon vacuum from the outset [19]. In addition, the universal solution can be expressed by a state in the universal Fock space [17]. These facts are required for the tachyon vacuum solution. If we can reproduce purely closed string amplitudes in the theory expanded around the universal solution, we have further evidence for the correspondence between the tachyon vacuum and the universal solution.

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